Periodic Control for Minimum-Fuel Aircraft Trajectories

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The problem of minimizing fuel consumption of an aircraft is formulated as an optimal control problem with periodic boundary conditions. Two problems are considered: one with constant aircraft weight and one with variable weight. Numerical solutions are computed via a multiple-shooting method and consist of bang-bang control actions for power setting. In comparison to the steady-state solutions, savings in fuel consumption for an F-4-type aircraft are approximately 2%. The solution obtained is shown to satisfy the second-order sufficiency conditions for a weak local optimum.

Introduction

THE problem of maximizing the range of a jet-powered aircraft for a given amount of fuel, considering oscillatory cruise paths, was first analyzed in Ref. 1. There it was concluded that an increase in range on the order of 10% might sometimes be possible; the physical explanation being that the speed for maximum engine efficiency is usually different from the speed for minimum drag. The problem of minimizing the amount of fuel used for a given distance was analyzed in Ref. 2, and it was shown that the singular arc corresponding to the steady-state cruise is nonminimizing since the generalized Legendre-Clebsch condition due to Kelley[3] is not satisfied. Some ideas regarding methods to improve fuel economy by periodic control were described in Ref. 4. In Ref. 5, a general theory for periodic processes was presented and applied to trajectory problems in Ref. 6.

In Ref. 7 and in this paper, a numerical solution of a periodic orbit of a realistically modeled aircraft is presented. The problem is addressed by minimizing the amount of fuel used per orbit in order to obtain the fuel savings possible in comparison to the steady-state solution. In addition, the effect of decreasing aircraft weight is investigated. To generate accurate solutions the multiple-shooting method presented in Ref. 8 and the associated computer code documented in Ref. 9 are used. The paper is divided into three parts. Part 1 contains the problem, together with all of the data necessary for the model used. Part 2 presents the steady-state solutions, and part 3 gives periodic solutions.

Problem Statement

Equations of Motion

The equations of motion of an aircraft flying in the vertical plane are

$$\frac{dV}{dx} = g[(\delta^* T(h,M) - D(h,M,C_L))/W - \sin\gamma]/(V\cos\gamma) \quad (1)$$

$$\frac{d\gamma}{dx} = g[L(h,M,C_L)/(W\cos\gamma) - 1]/V^2 \quad (2)$$

$$\frac{dh}{dx} = = \tan\gamma \quad (3)$$

$$\frac{dW}{dx} = -\delta^* T(h,M)^* c(h,M)/(V\cos\gamma) \quad (4)$$

Here, x is the range taken as an independent variable; V (velocity), γ (flight path angle), h (altitude) and W (weight) are the state variables; the lift coefficient C_L and power setting δ are the control variables. g is the gravitational acceleration, assumed to be constant ($=9.80665$ m/s^2). With air density ρ and speed of sound a given by

$$\rho(h) = 1.225^*\exp(-1.0228055 - 0.12122693\,\bar{h} + r)$$

with

$$\bar{h} = h/1000, \quad r = 1.0228055\exp(-\text{Arg})$$

and

$$\text{Arg} = -3.48643241E-2\,\bar{h} + 3.50991865E-3\,\bar{h}^2 - 8.33000535E-5\,\bar{h}^3 + 1.15219733E-6\bar{h}^4$$

$$a(h) = 20.0468\sqrt{\text{Temp}}$$

with

$$\text{Temp} = 292.1 - 8.877433E-3\,h + 0.193315E-6\,h^2 + 3.72E-12\,h^3$$

and the Mach number $M = V/a(h)$, lift L, drag D, maximal thrust T, and specific fuel consumption c are given as functions of h, M, and C_L:

$$L = \rho(h)SV^2C_L/2$$

$$D = \rho(h)SV^2(C_{DO}(M) + \Delta C_D(M,C_L))/2$$

$$C_{DO} = a_1\arctan(a_2(M - a_3)) + a_4$$

with

$$a_1 = 0.007381240246, \qquad a_2 = 34.165985747,$$

$$a_3 = 0.953467778, \qquad a_4 = 0.0323999753,$$

$$\Delta C_D = b_1C_L^2 + b_2C_L^4 + b_3C_L^6 + b_4C_L^8$$

$$T = (c_1 + c_2M + c_3M^2 + c_4M^3)\times 2\times g$$

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$$c = (d_1 + d_2 M + d_3 M^2 + d_4 M^3)/4263.84$$

with $S = 49.25\ \mathrm{m}^2$ as the reference area. These data represent an F-4 aircraft; the values for the polynomial coefficients are given in Table 1. x and h are measured in m; the unit for all forces is Newton.

Cost Functions and Boundary Conditions

Problem P1 consists of minimizing

$$I(\delta, C_L) = -W(x_f) \tag{5}$$

with boundary conditions

$$\begin{aligned} V(0) &= V(x_f) \\ \gamma(0) &= \gamma(x_f) \\ h(0) &= h(x_f) \\ W(0) &= W_0 \end{aligned} \tag{6}$$

and the final range x_f prescribed. P1 is not a periodic optimal control problem since weight is monotonically decreasing during flight. The control constraints are

$$0 \le C_L \le C_{L\max}, \qquad \delta_{\min} \le \delta \le 1 \tag{7}$$

Problem P2 consists of minimizing

$$I(\delta, C_L) = (W_0 - W(x_f))/x_f \tag{8}$$

by neglecting weight changes, i.e., by replacing W by W_0 in Eqs. (1) and (2) and by associating Eq. (4) with fuel used during flight. In contrast to P1, the range x_f is free; i.e., x_f is an object of the optimization process as well. P2 is truly a periodic optimal control problem.

Optimality Conditions

The Hamiltonian is defined as

$$H = \lambda_v g((\delta T - D)/W - \sin\gamma)/(V\cos\gamma) + \lambda_\gamma g(L/(W\cos\gamma) - 1)/V^2 + \lambda_h \tan\gamma - \lambda_w \delta T c/(V\cos\gamma) \tag{9}$$

The Euler-Lagrange equations are

$$\frac{d\lambda_v}{dx} = g\left[\lambda_v\left\{((\delta T - D)/W - \sin\gamma)/V - \left(\frac{\delta\partial T}{\partial V} - \frac{\partial D}{\partial V}\right)\Big/W\right\} - 2\lambda_\gamma\cos\gamma/V^2 - \delta\lambda_w/g\left(Tc/V - \frac{c\partial T}{\partial V} - \frac{T\partial c}{\partial V}\right)\right]\Big/(V\cos\gamma) \tag{10}$$

$$\frac{d\lambda_\gamma}{dx} = -[\lambda_v g(\sin\gamma(\delta T - D)/W - 1) + \lambda_\gamma g L\sin\gamma/(VW) + \lambda_h V - \lambda_w \delta T c\sin\gamma]/(V\cos^2\gamma) \tag{11}$$

$$\frac{d\lambda_h}{dx} = -\left[\lambda_v g\left(\frac{\delta\partial T}{\partial h} - \frac{\partial D}{\partial h}\right)\Big/W + \lambda_\gamma g L\frac{\partial\rho}{\partial h}\Big/(\rho V W) - \lambda_w \delta\left(\frac{c\partial T}{\partial h} + \frac{T\partial c}{\partial h}\right)\right]\Big/(V\cos\gamma) \tag{12}$$

$$\frac{d\lambda_w}{dx} = g(\lambda_v(\delta T - D) + \lambda_\gamma L/V)/(W^2 V\cos\gamma), \quad \text{for P1}$$
$$= 0, \qquad \text{for P2} \tag{13}$$

For both problems, C_L^* is obtained from

$$0 = \lambda_\gamma - V\lambda_v\frac{\partial\Delta C_D}{\partial C_L}, \quad 0 \le -\lambda_v\frac{\partial^2 D}{\partial^2 C_L} \tag{14}$$

and the optimal power setting δ^*,

$$\delta^* = \delta_{\min} \quad \text{if} \quad S > 0 \tag{15}$$

$$\delta^* = 1, \quad \text{if} \quad S < 0 \tag{16}$$

$$\delta_{\min} \le \delta^*(x) \le 1 \text{ if } S = 0 \text{ (singular control)} \tag{17}$$

with

$$S = \lambda_v g - \lambda_w W c \tag{18}$$

For P1 the transversality conditions are

$$\lambda_v(0) = \lambda_v(x_f), \qquad \lambda_\gamma(0) = \lambda_\gamma(x_f)$$
$$\lambda_h(0) = \lambda_h(x_f), \qquad \lambda_w(x_f) = -1 \tag{19}$$

Table 1 Polynomial coefficients for aircraft model

	Coefficients for drag polynomials $\Delta C_D(M, C_L)$			
	M^0	M	M^2	M^3
b_1	+0.21702097	−0.35536702	+0.36538446	−0.05068006
b_2	+0.29316475	−0.70255160	+0.66298838	−0.10156934
b_3	−0.20273272	+0.84738134	−0.60753806	+0.00153550
b_4	+0.03071373	−0.23330516	+0.12615549	+0.02759661
	Coefficients for thrust polynomials $T(M,h)$			
	h^0	h	h^2	h^3
c_1	+0.4304853E4	−0.38000872	+0.1004360E−4	−0.6250142E−10
c_2	+0.1922648E4	−0.73011970	+0.5187934E−4	−0.10380902E−8
c_3	−0.9983243E3	+0.1143555E1	−0.9524995E−4	+0.2094355E−8
c_4	−0.1314679E3	−0.29088446	+0.2665875E−4	−0.6097151E−9
	Coefficients for fuel flow polynomials $c(M,h)$			
	h^0	h	h^2	
d_1	+0.1232706E1	−0.1071711E−3	−0.7804461E−8	
d_2	−0.7570441E−1	+0.1858330E−3	−0.1270511E−7	
d_3	+0.3386447	−0.1348872E−3	+0.7411827E−8	
d_4	−0.9252903E−1	+0.3134176E−4	−0.1385409E−8	

For P2 one has, in addition

$$H(x_f) = I = (W_0 - W(x_f))/x_f \tag{20}$$

In Eqs. (9-12) and (18) W is replaced by W_0 in the case of P2. The problem thus consists of finding optimal controls $C_L^*(x)$ and $\delta^*(x)$ according to Eqs. (14-18) to solve the two-point boundary-value problem (TPBVP) consisting of the differential equations (1-4) and (10-13), and the boundary conditions (6), (19), and (20).

Steady-State solutions

Problem P2

A steady-state solution of the TPBVP can be associated with problem P2:

$$\gamma(x) = 0, \qquad L = W_0, \qquad \delta T = D \tag{21}$$

Equation (4) can be integrated, obtaining for the functional equation (8)

$$I(h, V) = Dc/V$$

A stationary value of I must satisfy

$$\frac{\partial I}{\partial V} = 0, \qquad \frac{\partial I}{\partial h} = 0$$

Using Newton's method the numerical values for our airplane model are, with $W_0 = 166{,}713$ N,

$$V_{ss} = 250.7886 \text{ m/s}, \qquad h_{ss} = 11196.8610 \text{ m}$$

$$\delta_{ss} = 0.6243, \qquad C_{Lss} = 0.3243$$

The Euler-Lagrange multipliers are

$$\lambda_{Vss} = -0.5027, \quad \lambda_{\gamma ss} = -14.0523, \quad \lambda_{hss} = -0.01966 \tag{22}$$

The value of the cost functional is

$$I_{ss} = 2.3847 \text{ kg/km}$$

The values satisfy all necessary conditions, and the control $\delta^*(x)$ is a totally singular control.

Problem P1

There is also a steady-state solution associated with problem P1. With Eq. (21) one has, from Eqs. (1-3),

$$\frac{dV}{dx} = 0, \qquad \frac{dh}{dx} = 0 \tag{23}$$

and with Eq. (4),

$$\frac{dW}{dx} = -D(V,h,W)\, c(h,M)/V \tag{24}$$

For given x_f, V and h are to be determined such that

$$I = -W(x_f, V, h)$$

is minimized. The necessary conditions for a minimum are

$$W_v(x_f,V,h) = \frac{\partial W}{\partial V} = 0, \qquad W_h(x_f,V,h) = \frac{\partial W}{\partial h} = 0 \tag{25}$$

Since $W(0,V,h) = W_0$ for all V and h, it must be true that

$$W_v(0,V,h) = 0, \qquad W_h(0,V,h) = 0 \tag{26}$$

Because of this relation, Eq. (25) can be expressed as

$$0 = W_v(x_f,V,h) - W_v(0,V,h)$$

$$= \int_0^{x_f} W_{vx}(x,V,h)\,dx = \int_0^{x_f} \frac{\partial(-Dc/V)}{\partial V} dx$$

and

$$0 = \int_0^{x_f} \frac{\partial(-Dc/V)}{\partial h} dx$$

leading to

$$\frac{dW_v}{dx} = Dc/V^2 - \{(D_v + D_L W_v)c + Dc_v\}/V$$

$$\frac{dW_h}{dx} = -\{(D_h + D_L W_h)c + Dc_h\}/V \tag{27}$$

The steady-state solution for P1 is obtained by solving the TPBVP consisting of differential equations (23), (24), and (27), with boundary conditions $W(0,V,h) = W_0$, (25) and (26), in the parameters V,h, and the variables $W(x,V,h)$, $W_v(x,V,h)$, $W_h(x,V,h)$ for given x_f. Solutions of this TPBVP are given for various x_f in Fig. 1.

These solutions are only feasible. They are not stationary solutions; that is, they do not satisfy the optimality conditions (10-19). An alternative flight program for the steady-state solution could be V and $C_L = $ const, leading to a different TPBVP, whose solution is the cruise-climb. Since we are interested in periodic boundary conditions, the steady-state cruise-climb solution is not a valid candidate for comparison.

Periodic Solutions

Fuel-Minimizing Solutions with Constant Weight

The assumption of constant weight implies that the solution of P2 is periodic: V, γ, h, λ_v, λ_γ, and λ_h are periodic, λ_W is constant, and the nonperiodic variable W does not affect the trajectory since it is used only to compute the cost functional. Figure 2 depicts a characteristic property of a periodic solution: The interval with length x_f can be shifted arbitrarily along a perodic solution, maintaining periodicity. Since the value of the switching function [Eq. (18)] must be the same at both ends of the interval, bang-bang solutions can have even

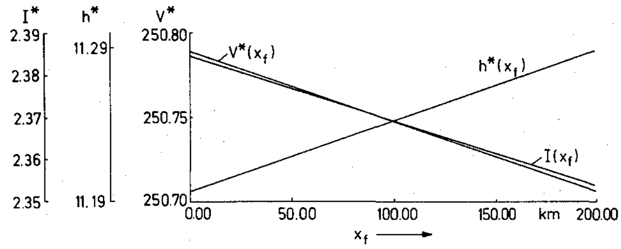

Fig. 1 Quasistationary solution with decreasing weight.

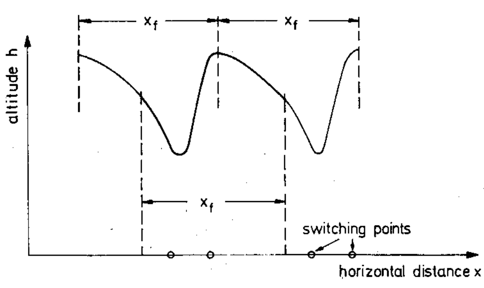

Fig. 2 Characteristics of a periodic solution.

numbers of switching points only. Because of the "shifting property" it can be assumed that $\delta = \delta_{min}$ and $V(0) = V_0$ at $x = x_0 = 0$, where V_0 is a prescribed initial value which is really passed in the minimal thrust phase of the solution.

Numerical solutions are obtained with the multiple-shooting technique described in Refs. 7-9. This algorithm is an extension of the ordinary multiple-shooting method in the sense that it was developed for the solution of "boundary-value problems with switching conditions." In addition to the usual two-point boundary conditions, this type of problem requires certain conditions to be satisfied at a priori unknown switching points. For instance, this kind of situation results from the optimality conditions, Eqs. (9)-(20), if a fixed switching structure is assumed. In this case, a bang-bang structure with two switching points (determined by the condition $S = 0$) was predicted, motivated by the consideration above. The algorithm, however, only guarantees that the equation $S = 0$ is satisfied at the switching points. It does not ensure that S has the correct sign within each subarc. This has to be verified afterwards for each solution candidate produced by the numerical method.

The problem turned out to be very critically dependent on the initial values for the costates. For instance, it is not possible to generate a bang-bang trajectory starting with the initial values, Eq. (22), although they do not differ much from the solution values. A large number of initial value problems had to be solved until a starting trajectory with the above-described bang-bang structure could be found at all. It started from the intial state

$$V_0 = 250\,\text{m/s}, \quad \gamma_0 = 1.1\,\text{deg}, \quad h_0 = 11{,}600\,\text{m}$$

reaching a final state $(V_1, \gamma_1, h_1) \neq (V_0, \gamma_0, h_0)$ after a range of about 50 km. This trajectory could still not serve as the initial value for the original problem. Instead it had to be varied cautiously within a long series of continuation steps which finally arrived at the solution of problem P2.

At first a continuation on the terminal state from (V_1, γ_1, h_1) to (V_0, γ_0, h_0) was performed neglecting the periodicity of the costate variables. Subsequently, the fixed initial (and terminal) velocity V_0 was varied, until one of the Lagrange multipliers reached the same value at both ends. By changing γ_0 and h_0, the periodicity of the two other λs could be achieved similarly. Up to this point the range x_f was kept constant. A last continuation series along x_f maintaining periodicity in V, γ, h, and their costates led to the optimal range x_f determined by the condition $H = I$. The solution data are given in Table 2 for $\delta_{min} = 0$.

At $x = x_1$ the power setting is switched from δ_{min} to $\delta = 1$; at $x = x_2$ it is switched back to δ_{min}, where x_f is the optimal periodic range and I the optimal fuel consumption. The solution is accurate to at least six digits. The fuel saving compared to the steady state is about 2%.

Figure 3 shows altitude, velocity, and lift coefficient as functions of normalized range for various values of δ_{min}. Clearly the length of the subinterval with $\delta = 1$ increases with decreasing δ_{min}. (The switching points are the points of discontinuous slopes in the V history.)

In order to prove the local optimality of the obtained solutions, the second-order sufficiency conditions for optimal periodic processes presented in Ref. 6 are applied. Defining

$$y(x) = (V, \gamma, h, \lambda_v, \lambda_\gamma, \lambda_h)^T$$

$$u(x) = (C_L, \delta)^T$$

and assuming that $u(x)$ is expressed through Eq. (14) as $u(x) = u(y(x))$, the differential system can be written as

$$\frac{dy}{dx} = K \frac{\partial H(y, u(y))}{\partial y} \tag{28}$$

where

$$K = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$$

is a 6×6 matrix, called the fundamental symplectic matrix in Ref. 5. The linearized equation associated with Eq. (28) can be written as

$$\frac{d\Delta y}{dx} = KH_{yy}\Delta y \tag{29}$$

where Δy is a deviation from the optimal path. The Jacobian matrix KH_{yy} can be written as

$$KH_{yy} = \begin{bmatrix} A & -B \\ C & -A^T \end{bmatrix}$$

where A, B, and C are 3×3 submatrices obtained from differentiating the right-hand sides of Eqs. (1-3) and (10-12) with

Table 2 Solution for problem P2

V, m/s λ_v x_1/x_f I, kg/km	γ, rad λ_γ x_2/x_f	h, m λ_h x_f, m
0.235252254E+03	0.837709049E−11	0.116440493E+05
−0.482397164	−0.128332358E+02	−0.199120502E−01
0.396743300	0.936401758	0.527933805E+05
0.2338058E1		

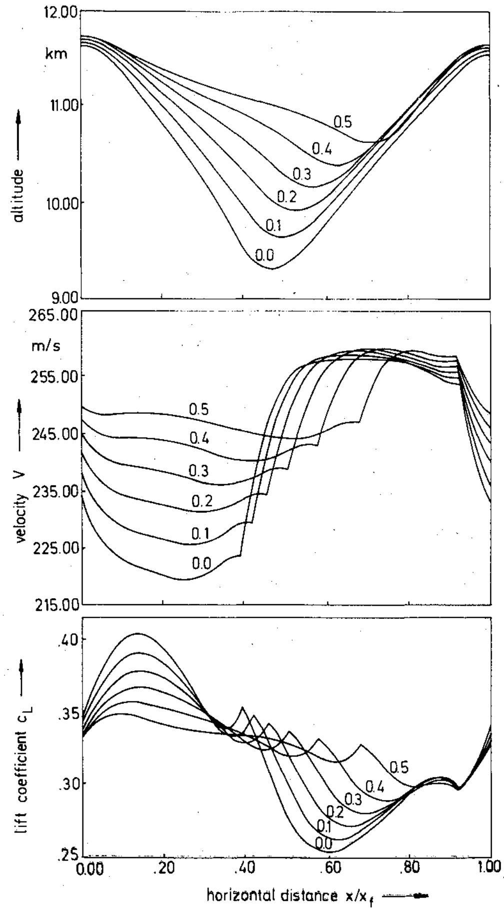

Fig. 3 Periodic solution for various minimum power settings.

respect to y. These expressions can be obtained easily and explicitly. The symmetric matrix Riccati differential equation associated with Eq. (29) is

$$\frac{dP}{dx} = -PA - A^T P + PBP + C \tag{30}$$

The transition matrix generated by

$$\frac{d\Phi(x,0)}{dx} = KH_{yy}\Phi(x,0), \qquad \Phi(0,0) = I \tag{31}$$

and evaluated at $x = x_f$ is called the monodromy matrix. $\Phi(x,0)$ is partitioned as

$$\Phi(x,0) = \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{bmatrix}$$

and the solution of Eq. (30) can be written as a function of $P(0)$ (see Ref. 6):

$$P(x) = (\Phi_{21}(x) + \Phi_{22}(x)P(0))(\Phi_{11}(x) + \Phi_{12}(x)P(0))^{-1} \tag{32}$$

The periodicity requirement $P(0) = P(x_f)$ yields an algebraic equation system for $P(0)$:

$$P(0) = (\Phi_{21}(x_f) + \Phi_{22}(x_f)P(0))(\Phi_{11}(x_f) + \Phi_{12}(x_f)P(0))^{-1} \tag{33}$$

The sufficiency conditions were verified as follows:

1) The initial value $P(0)$ was determined numerically from Eq. (33). Using this initial value, det $(\Phi_{11}(x) + \Phi_{12}(x)P(0))$ was found to be nonzero along the whole orbit. According to Eq. (32), this guarantees the existence of the matrix function $P(x)$.

2) The second derivatives of the cost function with respect to the switching points are positive. This could also be checked numerically. (For the computation of these quantities see Ref. 6.)

3) The strengthened Legendre-Clebsch condition for C_L is satisfied.

4) $\partial x_f / \partial H | H = J \neq 0$ was checked numerically by generating adjacent periodic orbits.

5) The eigenvalues of the monodromy matrix are distinct—except for the one at unity—and off the unit circle. In addition, if z is an eigenvalue so is $1/z$.

Summarizing this analysis, it has been confirmed by numerical computation that the periodic solution given above satisfies the sufficiency condition given in Ref. 5.

Moreover, the results under item 5 above again reflected the numerical ill-conditioning of the problem. A deviation $\Delta y(0)$ from the optimal initial value $y(0)$ is to first order propagated by

$$\Delta y(x) = \Phi(x,0)\Delta y(0) \tag{34}$$

[see Eqs. (29) and (31)]. This yields the estimation

$$\frac{\|\Delta y(x_f)\|}{\|\Delta y(0)\|} \leq \max_{z \in R^6 \setminus \{0\}} \frac{\|\Phi(x_f,0)z\|}{\|z\|} = \|\Phi(x_f,0)\| \tag{35}$$

The last expression, the norm of the monodromy matrix, obviously provides a quantitative measure for the sensitivity of system (28) with respect to the initial value. This norm could be estimated with the help of the eigenvalues computed under item 5 above:

$$\|\Phi(x_f,0)\| \geq \max\{ |z| \mid z \text{ eigenvalue of } \Phi(x_f,0)\} = 0.25E+7 \tag{36}$$

Finally, two experiments with singular subarcs should be mentioned.

1) It was examined whether an entirely singular control leading away from the constant height and velocity course could improve the steady state. The result, however, was negative.

2) An effort was made to modify the bang-bang solution by inserting singular subarcs at the switching points. The multiple-shooting algorithm, however, could not satisfy the switching conditions at the transition from the singular control to maximum (minimum) thrust.

These two negative results confirm the assumption that the presented bang-bang strategy is really optimal.

Fuel-Minimizing Solutions with Decreasing Weight

The optimal solution of problem P1 is no longer periodic due to the dependence of the right-hand sides of Eq. (1-4) and (10-13) on $W(x)$ and $\lambda_w(x)$. The trajectory belonging to P1 is not "shiftable," as demonstrated for the periodic solution in Fig. 2; the initial values $V(0)$, $\gamma(0)$, and $h(0)$ are uniquely determined by the problem itself. On the other hand, it can be expected that the loss of weight affects the solution only slightly. That means that the solution of P1 will be adjacent to a periodic solution of P2. These two considerations motivate the following strategy to solve P1:

Subproblem P1a defined by:

Minimize $W(x_f)$ with the boundary conditions

$$V(0) = V(x_f) = V_0$$

$$\gamma(0) = \gamma(x_f) = \gamma_0$$

$$h(0) = h(x_f) = h_0$$

was solved for a series of prescribed values (V_0, γ_0, h_0). Simultaneously, a solution of the original problem (P1) was attempted. Convergence for this case, however, could be achieved only at four initial conditions. Figure 4 shows the cost functional $I = (W_0 - W(x_f))/x_f$ of the solutions of P1a as functions of the abscissa $x/x_{f\text{opt}}$, where the initial values were taken from the periodic trajectory. The four points near this curve belong to the four solution candidates obtained for P1. Clearly the third point represents the optimal solution of P1. The others have a greater performance index and, moreover, violate the necessary condition of Eqs. (15) and (16) since the switching function changes its sign within the maximum or minimum thrust phase. Figure 5 depicts a comparison of solution three and the appropriate solution of P2, which is shifted such that the first thrust switching is identical.

For ranges $x_f > x_{f\text{opt}}$ the periodic process (problem P2) can be repeated, thus leading to a sequence of "power on"—"power off" control actions. Similarly, when the fixed range of problem P1 is enlarged, there is a point between $x_{f\text{opt}}$

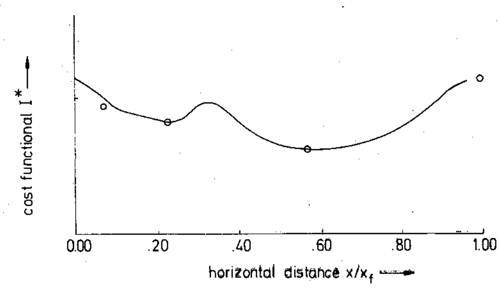

Fig. 4 Cost functional $I = (W_0 - W(x_f))/x_f$ as a function of initial conditions.

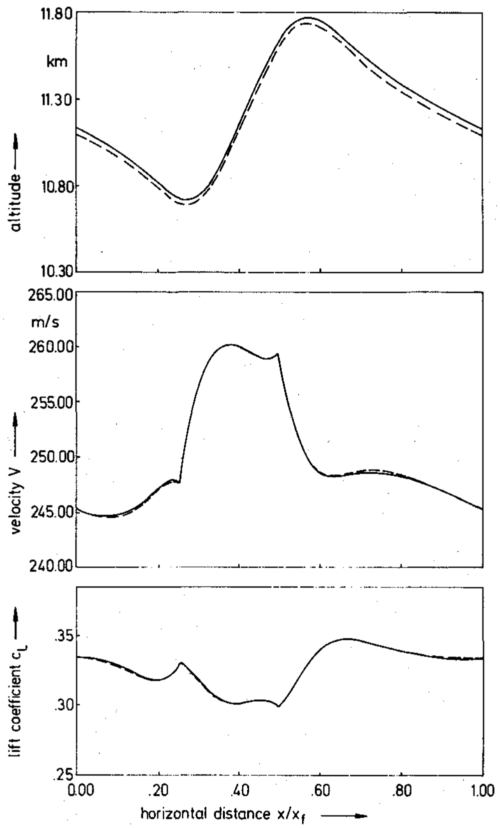

Fig. 5 Periodic solution with decreasing weight ($\delta_{min} = 0.5$).

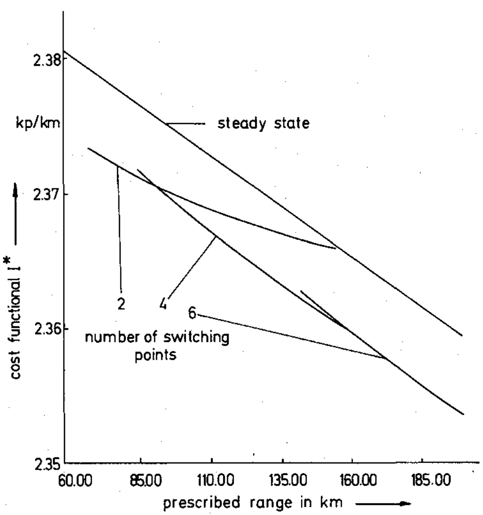

Fig. 6 Switching structure of periodic solutions as a function of prescribed range in comparison to steady-state solutions.

and $2^*x_{f\mathrm{opt}}$ where the optimal structure changes from 2 to 4 switching points. Exceeding $2^*x_{f\mathrm{opt}}$ there is a range were six switches become optimal, and so on. Therefore, solutions of P1 with varying range and switching structure were computed. Figure 6 depicts how the optimal strategy, dependent on the prescribed range, has to be selected. The same picture also contains the history of the cost functional of the steady solutions for P1 [Eqs. (23-27)]. It is observed that the relative amount of fuel saved by optimal control is nearly constant as a function of range.

Conclusions

This paper presents fuel-minimizing flight paths for a realistic aircraft model. When weight is assumed to be constant, the optimization task is an optimal periodic control problem. The optimal controls were computed by solving multipoint boundary-value problems derived from variational calculus. This solution was carried out with the greatest possible accuracy available in today's numerical methods.

Since power setting was modeled as a linear control, the optimal switching structure had to be found. In the periodic case this is a bang-bang strategy with two switching points. From further investigations it can be assumed that the optimal trajectory does not contain singular subarcs. Moreover, the obtained bang-bang solution could be proved to be locally optimal. This analysis again revealed the numerical ill-conditioning of the problem, which caused serious difficulties in the solution process.

The difference between constant and variable weight is primarily of theoretical nature. The practical effect of decreasing weight was found to be small.

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